Abstract

In this paper, we consider the problem of ranking linear budget sets with different available goods. We introduce axioms that are based on preference-based and preference-independent views of evaluating freedom, as well as two basic axioms. By using these axioms, we characterize two ranking rules.

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1. Introduction

In this paper, we consider the problem of ranking linear budget sets. Past studies of this problem (Xu, 2004; Kolm, 2004; 2010; Miyagishima, 2010) focused on the case where the set of goods is fixed. We consider the problem of evaluating linear budget sets with the possibility of different available goods. Our model can be applied, for instance, to the following problems: (1) evaluation of the degrees of freedom enjoyed by an agent at different points in time; and (2) comparison of the degrees of freedom...
enjoyed by agents in different countries.

We introduce axioms consistent with *preference-based* and *preference-independent* views of evaluating freedom. The axioms are associated with (i) cases where new goods become available and (ii) weights representing the importance of goods. “Preference-independent” means that the agent’s preferences do not play any role. In contrast, “preference-based” means that evaluation of freedom depends on the agent’s preferences. We explain these points below.

(i) **Axioms for new goods:**

We introduce two axioms for cases where new goods become available. First, the axiom named *Independence of New Goods with Infinite Price Level (IIP)* requires that if a new good becomes available with an infinitely high price level (in the limit), then the degree of freedom offered by the budget set does not change. If a ranking rule satisfies IIP and the axiom named *Monotonicity*, then new goods (with finite price levels) always increase the agent’s freedom. This fact means that whenever the set of available goods expands, the degree of freedom increases independently of the new goods’ price levels. Notice that the preference need not play any role in this argument. Therefore, IIP (combined with Monotonicity) is consistent with the preference-independent view.

Second, the axiom named *Cutoff Price Levels (CPL)* requires that each new good should have a fixed cutoff price level: when the new good’s price is at the cutoff level,

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This axiom (formally introduced in Section 3) requires that if new consumption bundles become available in a budget set, the degree of freedom offered by the budget set should increase.
the freedom does not change after the good becomes available. The cutoff level could be interpreted as determined by a kind of preference related to the quantity of the good the agent wants. If the agent desires to consume a large amount of the new good, the cutoff price level would be lower. It is easily shown that if a ranking rule satisfies Monotonicity and CPL, then the freedom offered by a budget set increases if and only if the price of the new good is lower than the critical price level. An intuitive explanation is as follows. Suppose a new good becomes available. Then, the agent desires that the new good’s price be low enough that he/she can buy a certain amount of the good and spend sufficiently on other goods. In this case, if the price level of the new good is low enough to be below the cutoff price level, the agent judges that the good increases freedom sufficiently to achieve a higher level of well-being than before.

In Sections 4 and 5, we give examples to show that IIP and CPL (combined with Monotonicity) are consistent with the preference-independent and the preference-based views, respectively.

(ii) Axioms for weights of goods:

We next introduce two axioms relevant to weights of goods. First, the axiom named Symmetry (introduced by Xu, 2004) requires that if the prices of any two goods are permuted, the freedom provided by the budget set should not change. This would imply that preference is irrelevant for the evaluation of freedom of choice. In this sense, this axiom is consistent with preference-independent evaluations of freedom.
Second, the axiom named *Weighting Goods* requires that each good should have a weight in terms of the agent’s well-being. The weights could be interpreted as determined by the agent’s preference for the goods. The higher the agent’s demand for a good, the higher the weight of the good. Thus, this axiom is consistent with preference-based evaluations of freedom.

In this paper, we propose two ranking rules. We show that these two rules are characterized by using the axioms above consistent with the preference-independent and preference-based views, respectively, as well as the two axioms satisfied commonly by the two rules.

We make two remarks. First, our analysis is based technically on the theory of the utilitarian social welfare function and population ethics in social choice theory. The reader may notice that our characterization results (Theorems 1 and 2) are similar to those of Blackorby et al. (2005, Theorem 5.5). Hence, our results show a technical relation between the theories of ranking linear budget sets and population issues in social choice theory.

Second, we interpret the cutoff price levels and weights of goods discussed above as depending on the agent’s preferences. However, it is difficult to actually derive the critical price levels and the weights endogenously. Therefore, we assume that the cutoff price levels and the weights of goods are exogenously given.

This paper is organized as follows. In Section 2, we present the basic notation
and definitions. In Section 3, we introduce two axioms that are commonly satisfied by
the two ranking rules. In Section 4, we characterize one ranking rule in terms of the
preference-independent view, and in Section 5, we characterize the other ranking rule
in terms of the preference-based view. In Section 6, we compare our study with others
in the literature. In Section 7, we provide some concluding remarks.

2. Notation

Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of possible goods. $\mathcal{N} = \{M \subseteq \mathbb{N} | M$ is finite and
nonempty} is the family of sets of available goods. For all $n \in \mathbb{N}$, let $\mathbb{R}_{++}^n = \mathbb{R}_{++}^n \cup \{\infty\}$
with the conventions that $\infty + a = \infty (\forall a \in \mathbb{R}_{++}), \infty \times a = \infty (\forall a \in \mathbb{R}_{++})$, and $1/\infty =
0$. For any $N \in \mathcal{N}$ and any $p_N = (p_i)_{i \in N} \in \mathbb{R}_{++}^{|N|}, B(p_N) = \{x \in \mathbb{R}_{++}^{|N|} | \sum_{i \in N} p_i x_i \leq 1\}$ is
a budget set. The family of budget sets is denoted by $\mathcal{B} \equiv \{B(p_N) | p_N \in \mathbb{R}_{++}^{|N|}, N \in \mathcal{N}\}$.

Let $\succeq$ be a ranking over $\mathcal{B}$ that is reflexive and transitive (but not necessarily
complete). For any $B(p_N), B(q_M) \in \mathcal{B}$, $[B(p_N) \succeq B(q_M)]$ means $[B(p_N)$ offers at least
as much freedom as $B(q_M)]. \succ$ and $\sim$ are, respectively, the asymmetric and symmetric
parts of $\succeq$.

3. Basic axioms

In this section, we introduce two axioms that are both satisfied by the two ranking
rules proposed in this paper.

Monotonicity: For any $B(p_N), B(q_N) \in \mathcal{B}$, if $B(q_N) \subset B(p_N)$ and $B(q_N) \neq B(p_N),$
then $B(p_N) \succ B(q_N)$. 

Independence of Equal Changes in Maximal Amounts (IEM): For any $B(p_N)$, $B(q_N)$, $B(p'_N)$, $B(q'_N) \in B$ and any $i \in N$, if $\frac{1}{p_i} = \frac{1}{p'_i} + \delta$ and $\frac{1}{q_i} = \frac{1}{q'_i} + \delta$ ($\delta \in \mathbb{R}$), and for all $j \neq i$, $p_j = p'_j$, and $q_j = q'_j$, then

$$B(p_N) \succeq B(q_N) \iff B(p'_N) \succeq B(q'_N).$$

Monotonicity requires that the freedom should increase when additional consumption bundles become available. This axiom is consistent with preference-independent evaluations of freedom, because the additional bundles can increase freedom irrelevant of preferences. This axiom is also compatible with preference-based evaluations, because additional commodity bundles could increase the possibility of higher well-being.

IEM requires that if the difference between the maximal amounts of good $i$ in two budget sets, $p_i^{-1} - q_i^{-1}$, remains invariant after a price change, then the ranking over the two budget sets should not change. Note that the difference between the maximal amounts of goods may give us information concerning freedom in terms of both the preference-independent and the preference-based views. This is because these two would be relevant to the amounts of goods that are available. Hence, IEM claims that ranking rules should compare budget sets consistently with such information.

4. A ranking based on the preference-independent view

In this section, we characterize a ranking rule from the preference-independent viewpoint. We first introduce two axioms.
**Symmetry:** For any $B(p_N) \in \mathcal{B}$ such that $|N| \geq 2$, and any $i, j \in N$,

$$B(p_i, p_j, p_{N\setminus\{i,j\}}) \sim B(p_j, p_i, p_{N\setminus\{i,j\}}).$$

*Symmetry* requires that if the prices of two goods are permuted, the freedom offered by the budget set should not change. This axiom is proposed by Xu (2004) as a principle that there is *no discrimination* among goods when comparing budget sets in terms of freedom. An important implication of this axiom is that preferences do not play any role in evaluations of freedom.

Next, we introduce the axiom for new goods.

**Independence of New Goods with Infinite Price Level (IIP):** For any $B(p_N) \in \mathcal{B}$, and any $k \notin N$,

$$B(p_N) \sim B(p_N, \infty_k),$$

where $B(p_N, \infty_k) = \lim_{p_k \to \infty} B(p_N, p_k)$.

*IIP* insists that if a new good appears with an infinitely high price, then the freedom of the budget set does not change. We discuss the consistency between this axiom and *preference-independent* evaluations of freedom, by using the following example. Suppose that a ranking rule satisfies *IIP* and *Monotonicity*. Consider a linear budget set $B(p_N)$. Let $k$ be a new good. Then, by *IIP*, $B(p_N) \sim B(p_N, \infty_k)$. Moreover, by *Monotonicity*, $B(p_N, p_k) \succ B(p_N, \infty_k)$ for any $p_k < \infty$. Thus, by transitivity, we obtain $B(p_N, p_k) \succ B(p_N)$ for any $p_k < \infty$. For instance, let $k$ be a new medicine for an intractable disease. Suppose that the agent is not affected by the disease, and thus
does not seek to purchase the medicine. Nevertheless, because a new good has become available, the new budget set $B(p_N, p_k)$ provides more freedom than $B(p_N)$ (whenever $p_k$ is finite).¹ Note that preferences need not play any role in this argument. Therefore, this example would show that $IIP$ (combined with $Monotonicity$) is consistent with the preference-independent view.²

Our ranking rule characterized by the above axioms is defined as follows.

**Definition 1:** The ranking rule $\succeq_{PI}$ over $B$ is defined as follows:

$$\forall B(p_N), B(q_M) \in B, \ B(p_N) \succeq_{PI} B(q_M) \iff \sum_{i \in N} \frac{1}{p_i} \geq \sum_{i \in M} \frac{1}{q_i}. $$

This rule evaluates budget sets by the sum of the available amounts of goods (note that good $i$’s available amount is measured by $1/p_i$).

Our characterization of this ranking rule is as follows.

**Theorem 1:** Suppose $\succeq$ is reflexive and transitive. Then, $\succeq$ satisfies $Monotonicity$, $IEM$, $Symmetry$, and $IIP$ if and only if $\succeq = \succeq_{PI}$.

**Proof:** It is obvious that $\succeq_{PI}$ satisfies the axioms in the theorem. In the following, we show that if $\succeq$ satisfies the axioms, then $\succeq = \succeq_{PI}$.

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¹Note that this property is the same as that of social orderings satisfying *Existence of Critical Level* and *Strong Pareto* in the literature on population issues and social choice. See Blackorby et al. (2005).

²This axiom is also consistent with some preference-based evaluations. For instance, we could argue that in the above example, the freedom increased in terms of evaluation with a hypothetical preference that if an agent was affected by the disease, he/she would prefer to buy the medicine.
First, because $\succeq$ satisfies Monotonicity, IEM, and Symmetry, we can show the following fact:

$$\forall B(p_N), B(q_N) \in \mathcal{B}, \ B(p_N) \succeq B(q_N) \iff \sum_{i \in N} \frac{1}{p_i} \geq \sum_{i \in N} \frac{1}{q_i}.$$ (1)

When $|N| = 1$, this fact clearly holds by Monotonicity and reflexivity of $\succeq$. When $|N| \geq 2$, we prove (1) by using the methodology of d’Aspremont and Gevers (1977, Theorem 3).\(^6\) First, we show that

$$\forall B(p_N), B(q_N) \in \mathcal{B}, \ \sum_{i \in N} \frac{1}{p_i} = \sum_{i \in N} \frac{1}{q_i} \implies B(p_N) \sim B(q_N).$$ (2)

For convenience, we rewrite the budget sets as follows:

$$B(p_N) = b(P_N), \ B(q_N) = b(Q_N),$$

where $P_i = 1/p_i$ and $Q_i = 1/q_i$ for all $i \in N$. Next, let $b(P^0_N)$ (respectively $b(Q^0_N)$) be the budget set obtained by permuting the elements of $P_N$ (respectively $Q_N$) so that they are in order of increasing size. By Symmetry, $b(P_N) \sim b(P^0_N)$ (respectively $b(Q_N) \sim b(Q^0_N)$). Let $b(P^1_N)$ (respectively $b(Q^1_N)$) be the budget set obtained by subtracting $\min\{ P^0_i, Q^0_i \} (\forall i \in N)$ from $P^0_i$ (respectively $Q^0_i$). By IEM, $b(P^0_N) \sim b(Q^0_N)$ if $b(P^1_N) \sim b(Q^1_N)$.

By repeating this procedure a finite number $m$ of times, we have $b(P^m_N) = b(Q^m_N) = b(0, \ldots, 0)$. Thus, by reflexivity, $b(P^m_N) \sim b(Q^m_N)$. Moving back the chain of implications, by transitivity, we obtain $B(p_N) = b(P_N) \sim b(Q_N) = B(q_N)$.

\(^6\)Note that the rule in (1) is the same as the utilitarian social welfare function. The author thanks Toyotaka Sakai for commenting that the proof of d’Aspremont and Gevers (1977, Theorem 3) is applicable to that of Miyagishima (2010, Theorem 1), which is similar to (1).
Next, we show that
\[ \forall B(p_N), B(q_N) \in B, \sum_{i \in N} \frac{1}{p_i} > \sum_{i \in N} \frac{1}{q_i} \implies B(p_N) \succ B(q_N). \] (3)

Let \( B(p'_N), B(q'_N) \) be such that \( p'_i < q'_i \) for all \( i \in N \) and
\[ \sum_{i \in N} \frac{1}{p_i} = \sum_{i \in N} \frac{1}{p'_i}, \sum_{i \in N} \frac{1}{q_i} = \sum_{i \in N} \frac{1}{q'_i}. \]

Obviously, there exist such budget sets. Then, by (2), \( B(p_N) \sim B(p'_N) \) and \( B(q_N) \sim B(q'_N) \). By Monotonicity, \( B(p'_N) \succ B(q'_N) \). From transitivity, \( B(p_N) \succ B(q_N) \).

Combining (2) and (3), we have obtained (1).

Next, for convenience, let
\[ PI(p_N) = \sum_{i \in N} \frac{1}{p_i}. \]

We show that for any \( B(p_N), B(q_M) \in B, \)
\[ PI(p_N) \geq PI(q_M) \iff B(p_N) \succeq B(q_M). \]

By IIP,
\[ B(p_N, \infty_{M\setminus(N\cap M)}) \sim B(p_N), \] (4)
and
\[ B(\infty_{N\setminus(N\cap M)}, q_M) \sim B(q_M). \] (5)

Note that
\[ PI(p_N, \infty_{M\setminus(N\cap M)}) = \sum_{i \in N} \frac{1}{p_i} + [\lvert M \rvert - (\lvert N \cap M \rvert)] \times 0 = \sum_{i \in N} \frac{1}{p_i} = PI(p_N). \]
Similarly,
\[ PI(\infty_{N \setminus (N \cap M)}, q_M) = \sum_{i \in M} \frac{1}{q_i} + [ |N| - (|N \cap M|)] \times 0 = \sum_{i \in M} \frac{1}{q_i} = PI(q_M). \]

Thus, from (1),
\[ B(p_N, \infty_{M \setminus (N \cap M)}) \succeq B(\infty_{N \setminus (N \cap M)}, q_M) \iff PI(p_N) \geq PI(q_M). \]

By (4), (5), and transitivity,
\[ B(p_N) \succeq B(q_M) \iff \sum_{i \in N} \frac{1}{p_i} \geq \sum_{i \in M} \frac{1}{q_i}. \]

Therefore, we have obtained the desired result. \( \square \)

5. A ranking based on the preference-based view

In this section, we characterize a ranking rule from the preference-based viewpoint. First, we introduce the axiom related to weights of goods. As discussed in the Introduction, we assume that for any \( i \in N \), the weight of good \( i, t_i > 0 \), is given. We also assume that the weights are determined by a preference.

**Weighting Goods:** For any \( B(p_N), B(p'_N) \in B \) such that \( |N| \geq 2 \), and any \( i, j \in N \),

if \( t_i \frac{1}{p_i} = t_j \frac{1}{p_j} \) and \( t_j \frac{1}{p_j} = t_i \frac{1}{p_i} \), and \( p_k = p'_k \) for all \( k \neq i, j \), then \( B(p_N) \sim B(p'_N) \).

This axiom requires that for any two goods \( i \) and \( j \) in two budget sets \( B(p_N) \) and \( B(q_N) \), if the value of \( 1/p_i \) (respectively \( 1/p_j \)) evaluated by the weight of the good, \( t_i \) (respectively \( t_j \)), is equal to the value of \( 1/p'_j \) (respectively \( 1/p'_i \)) evaluated by \( t_j \).
and if the prices of other goods are the same, then the two budget sets should offer the same degree of freedom. In other words, this axiom insists that ranking rules should compare budget sets consistent with the weights of goods.

Next, we introduce the axiom named Cutoff Price Levels (CPL).

**Cutoff Price Levels (CPL):** For any $k \in \overline{N}$, there exists $p^*_k \in \mathbb{R}^+$ such that for all $B(p_N) \in B$ with $k \notin N$,

$$B(p_N, p^*_k) \sim B(p_N).$$

This axiom insists that the cutoff price level of a new good $k$ should exist where the presence of the good does not affect the degree of freedom. Note that the cutoff price levels possibly differ among goods because of differences in the importance of goods. Then, the cutoff level of each good would be determined by the agent’s preference in relation to amounts of the good he/she wants. We argue that CPL is consistent with the preference-based view of freedom. If a new good $k$ becomes available, by CPL, there exists $p^*_k$ such that $B(p_N) \sim B(p_N, p^*_k)$. Moreover, by Monotonicity, $B(p_N, p_k) \succ B(p_N, p^*_k)$ if and only if $p_k < p^*_k$. By transitivity, we obtain $B(p_N, p_k) \succ B(p_N)$ if and only if $p_k < p^*_k$. We give an intuitive explanation using the example of a new medicine discussed in Section 4. We assume here that the agent is affected by the intractable disease and thus needs the medicine. However, he/she has to buy other necessity goods in $N$ (for instance, food or clothes) in order to live. If the price of the medicine is too

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Note that when the weights are reference prices, then $t_i/p_i$ is the expenditure to buy $1/p_i$ units of good $i$ with price $t_i$. 7
high, he/she cannot purchase sufficient amounts of the other goods. In this case, \( p_k^* \) is relevant to the agent’s preferences. The agent desires that the price of the medicine is lower than \( p_k^* \). The larger the amount of medicine the agent needs, the smaller \( p_k^* \) would be. When the medicine becomes available with \( p_k < p_k^* \), it is judged that the agent’s freedom increases. If \( p_k^* \) is determined by the agent’s preference, CPL (combined with Monotonicity) is compatible with the preference-based view of evaluating freedom.

Now, we introduce the following parameterized family of rules.

**Definition 2:** Suppose that for any good \( i \in N \), parameters \( t_i > 0 \) and \( a_i \geq 0 \) are given. Then, the ranking rule \( \succeq_{PB}^{t,a} \) over \( B \) is defined as follows. \( \forall B(p_N), B(q_M) \in B \),

\[
B(p_N) \succeq_{PB}^{t,a} B(q_M) \iff \sum_{i \in N} t_i \left( \frac{1}{p_i} - a_i \right) \geq \sum_{i \in M} t_i \left( \frac{1}{q_i} - a_i \right).
\]

\( a_i \) can be interpreted as the amount of good \( i \) that should be secured for an agent. This rule evaluates a budget set by the weighted sum of the differences between the maximal amounts and the secured amounts of goods.

Following Miyagishima (2010, Section 5), it can be shown that the rule \( \succeq_{PB}^{t,a} \) is independent of rescaling units of goods if the weights are appropriately rescaled. For instance, if the weight of each good \( i, t_i \), is considered the reference price level, then the rankings by \( \succeq_{PB}^{t,a} \) are invariant to rescaling of the units.

We show the following characterization result.

**Theorem 2:** Suppose that for any good \( i \in N \), the weight \( t_i > 0 \) is given. \( \succeq \) satisfies Monotonicity, IEM, Weighting Goods, and CPL if and only if there exists \( a_i \) for
each $i$ and $\succeq_{PB}^t \succeq_{PB}^{t,a}$.

**Proof:** It is obvious that $\succeq_{BP}^{t,a}$ satisfies the axioms in Theorem 2. We show below that if $\succeq$ satisfies the axioms, then $\succeq = \succeq_{BP}^{t,a}$.

First, we show that if $\succeq$ satisfies *Monotonicity*, *IEM*, and *Weighting Goods*, then for any $B(p_N), B(q_N) \in \mathcal{B}$,

$$B(p_N) \succeq B(q_N) \iff \sum_{i \in N} t_i \frac{1}{p_i} \geq \sum_{i \in N} t_i \frac{1}{q_i}. \quad (6)$$

When $|N| = 1$, this fact clearly holds by *Monotonicity* and reflexivity of $\succeq$. When $|N| \geq 2$, we prove (6) by following the same discussion as (1). First, we show that

$$\forall B(p_N), B(q_N) \in \mathcal{B}, \sum_{i \in N} t_i \frac{1}{p_i} = \sum_{i \in N} t_i \frac{1}{q_i} \implies B(p_N) \sim B(q_N). \quad (7)$$

For convenience, we rewrite the budget sets as follows:

$$B(p_N) = b(P_N), \ B(q_N) = b(Q_N),$$

where $P_i = 1/p_i$ and $Q_i = 1/q_i$ for all $i \in N$. Next, let $b(P_N^0)$ (respectively $b(Q_N^0)$) be the budget set such that $P_i^0 = t_{\pi(i)} P_{\pi(i)}/t_i$ (respectively $Q_i^0 = t_{\pi'(i)} Q_{\pi'(i)}/t_i$) for a bijection $\pi : N \to N$ (respectively $\pi' : N \to N$) and for $i, j \in N$ with $i < j$, $P_i^0 < P_j^0$. By *Weighting Goods*, $b(P_N) \sim b(P_N^0)$ (respectively $b(Q_N) \sim b(Q_N^0)$). Let $b(P_N^1)$ (respectively $b(Q_N^1)$) be the budget set obtained by subtracting $\min\{P_i^0, Q_i^0\}$ ($\forall i \in N$) from $P_i^0$ (respectively $Q_i^0$). By *IEM*, $b(P_N^1) \sim b(Q_N^1)$ if $b(P_N^1) \sim b(Q_N^1)$. Let $b(P_N^m)$ (respectively $b(Q_N^m)$) be the budget set obtained by subtracting $\min\{P_i^0, Q_i^0\}$ ($\forall i \in N$) from $P_i^0$ (respectively $Q_i^0$). By *IEM*, $b(P_N^m) \sim b(Q_N^m)$ if $b(P_N^m) \sim b(Q_N^m)$.

By repeating this procedure a finite number $m$ of times, we have $b(P_N^m) = b(Q_N^m) = b(0, ..., 0)$. Thus, by reflexivity, $b(P_N^m) \sim b(Q_N^m)$. Moving back the chain of implications, by transitivity, we obtain $b(P_N) = B(p_N) \sim B(q_N) = b(Q_N)$. 

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Next, we show that
\[
\forall B(p_N), B(q_N) \in B, \sum_{i \in N} \frac{t_i}{p_i} > \sum_{i \in N} \frac{t_i}{q_i} \implies B(p_N) \succ B(q_N). \quad (8)
\]

Let \( B(p'_N), B(q'_n) \) be such that \( p'_i < q'_i \) for all \( i \in N \) and
\[
\sum_{i \in N} \frac{t_i}{p_i} = \sum_{i \in N} \frac{t_i}{p'_i}, \sum_{i \in N} \frac{t_i}{q_i} = \sum_{i \in N} \frac{t_i}{q'_i}.
\]

Obviously, there exist such budget sets. Then, by (2), \( B(p_N) \sim B(p'_N) \) and \( B(q_N) \sim B(q'_N) \). By Monotonicity, \( B(p'_N) \succ B(q'_N) \). From transitivity, \( B(p_N) \succ B(q_N) \).

Combining (7) and (8), we obtain (6).

Next, we show that for any \( B(p_N), B(q_M) \in B \),
\[
B(p_N) \succeq B(q_M) \iff \sum_{i \in N} t_i \left( \frac{1}{p_i} - a_i \right) \geq \sum_{i \in M} t_i \left( \frac{1}{q_i} - a_i \right).
\]

From CPL,
\[
B(p_N, p^*_M \setminus (N \cap M)) \sim B(p_N)
\]

and
\[
B(q^*_N \setminus (N \cap M), q_M) \sim B(q_M)
\]
hold. (Note that $CPL$ implies $p_i^* = q_i^*$ for any $i$.) Thus, (5) and transitivity of $\succeq$ imply

$$B(p_N) \succeq B(q_M) \iff B(p_N; p_{M \setminus (N \cap M)}) \succeq B(q_{N \setminus (N \cap M)}; q_M).$$

$$\iff \sum_{i \in N} t_i \frac{1}{p_i} + \sum_{i \in M \setminus (N \cap M)} t_i \frac{1}{p_i^*} \geq \sum_{i \in M} t_i \frac{1}{q_i} + \sum_{i \in N \setminus (N \cap M)} t_i \frac{1}{q_i^*}.$$

$$\iff \sum_{i \in N} t_i \frac{1}{p_i} - \sum_{i \in M \setminus (N \cap M)} t_i \frac{1}{p_i^*} \geq \sum_{i \in M} t_i \frac{1}{q_i} - \sum_{i \in N \setminus (N \cap M)} t_i \frac{1}{q_i^*}.$$

Let $\frac{1}{p_i^*} = a_i$ for all $i$. Then, we obtain $\succeq \succeq t, a$. $\square$

6. Discussion

Our rules are similar to Kolm’s (2010) criterion defined as follows: for any $B(p_N), B(q_N) \in B$,

$$B(p_N) \succeq B(q_N) \iff \sum_{i \in N} t_i p_i \leq \sum_{i \in N} t_i q_i.$$

In this section, we argue that the former may be more appropriate than an extended version of the latter when considering the problem in this paper. We also discuss a generalization of these rules.

An extension of Kolm’s rule to compare budget sets with different available goods is as follows. Given the weight $t_i$ for each good $i$, for all $B(p_N), B(q_M) \in B$,

$$B(p_N) \succeq B(q_M) \iff \sum_{i \in N} t_i p_i \leq \sum_{i \in M} t_i q_i.$$  \hspace{1cm} (9)

When using this rule, any new good always reduces the freedom, because for all $B(p_N)$
and \( p_k > 0 \) such that \( k \not\in N \),

\[
\sum_{i \in N} t_i p_i < \sum_{i \in N} t_i p_i + t_k p_k.
\]

Moreover, according to this rule, it is desirable that only one good remains in the market. These facts show that the rule would be inconsistent with freedom in the context of the present paper. In contrast, our rules would be useful in the context of this paper, as discussed above.

Notice that the forms of our rules and (9) are crucially relevant to IEM and \textit{Independence of Equal Price Changes (IEPC)}, respectively. IEPC is introduced by Miyagishima (2010) as one of the axioms that characterizes Kolm’s rule.

**Independence of Equal Price Changes (IEPC):** For all \( B(p_N), B(q_N) \in B \) such that \( |N| \geq 2 \), for all \( i \in N \), and for all \( \delta \in \mathbb{R} \) such that \( p_i + \delta > 0 \) and \( q_i + \delta > 0 \),

\[
B(p_N) \succeq B(q_N) \iff B(p_i + \delta, p_{N \setminus \{i\}}) \succeq B(q_i + \delta, q_{N \setminus \{i\}}).
\]

Although IEPC and IEM are similar, IEPC could lead to the problem discussed above.

Next, we give a general independence condition of which IEM and IEPC are special cases.

**\( \varphi \)-Independence:** Given a function \( \varphi_i : \mathbb{R}_+^+ \rightarrow \mathbb{R}_+ \) for all \( i \in N \), for all \( B(p_N), B(q_N), B(p'_N), B(q'_N) \in B \) and all \( j \in N \), if

\[
\varphi_j(p'_j) - \varphi_j(p_j) = \varphi_j(q'_j) - \varphi_j(q_j),
\]
and $p_k = p'_k$, $q_k = q'_k$ for all $k \neq j$, then
\[ B(p_N) \succeq B(q_N) \iff B(p'_N) \succeq B(q'_N). \]

This axiom requires a ranking rule to be independent of price changes such that the difference in prices transformed by $\phi_j$ does not change. This axiom obviously includes IEPC, IEM, and Xu’s (2004) Invariance of Scaling Effects (ISE) as special cases. In the cases of $\varphi_i(p) = -p$, $\varphi_i(p) = 1/p$, and $\varphi_i(p) = -\ln p$, the axiom is equivalent to IEPC, IEM, and ISE, respectively.

We introduce a ranking criterion, denoted by $\succeq_\phi$, satisfying $\phi$-Independence.

**Definition:** Given a function $\phi_i : \mathbb{R}_+^+ \to \mathbb{R}_+$ for all $i \in N$, for all $B(p_N), B(q_M) \in B$, and all $j \in N$,
\[ B(p_N) \succeq_\phi B(q_M) \iff \sum_{i \in N} \phi_i(p_i) \geq \sum_{i \in M} \phi_i(q_i). \]

This rule is a generalization of, for instance, $\succeq_{PI}$, $\succeq_{I,P^a}$, the volume criterion, and the criterion defined by (9). If $\varphi_i = \phi_i$ for all $i$, $\succeq_\phi$ satisfies $\phi$-Independence. Moreover, if $\phi_i$ is a decreasing function, $\succeq_\phi$ satisfies Monotonicity.

7. Concluding remarks

In this paper, we have proposed two ranking rules and characterized them in terms of the preference-independent and preference-based views of evaluating freedom. These ranking rules differ in two respects. First, the rule consistent with the preference-independent view does not weight goods, whereas the rule compatible with
the preference-based view respects the differences in weights among goods. Regarding this point, Symmetry and Weighting Goods separate the rules in terms of the two different viewpoints.

Second, when using the rule consistent with the preference-independent consideration, the freedom always increases if a new good becomes available. In contrast, the rule compatible with the preference-based view takes into account the minimum required amount of each good. In this respect, IH and CPL divide the two rules. In particular, the axiom CPL could be relevant to arguments on “necessaries” by classical economists. For instance, Smith (1776, p. 1168) stated the following in his discussion about necessaries: “By necessaries I understand not only the commodities that are indispensably necessary for the support of life, but also whatever the customs of the country renders it indecent for creditable people, even the lowest order to be without.” Sen (1999) also argued that it would be difficult to participate in social life without such necessaries. Of course, such necessaries would differ between different societies.

As mentioned in Section 5, \( \gtrsim^{t,a}_{PB} \) could be invariant to unit changes if the weights are also rescaled accordingly. In contrast, \( \gtrsim_{PI} \) is not independent of unit changes. In this respect, we can say that the former rule has a better property than the latter.

Finally, we mention two problems to be considered in future research. First, recall that in Theorem 2, the weight and the minimum amount of each good secured are fixed. However, the minimum amount secured and the weight may change over time and across countries. We have to construct a ranking rule taking into account this
point. Second, it remains for future research to axiomatize the rule defined by (9).

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